Supplementary Figure 1: Images of the parametric phase-locked oscillator. a, Optical image of the device. The \( \frac{\lambda}{4} \)-type coplanar waveguide resonator is made of 150-nm-thick niobium film deposited on an oxidized silicon substrate. b, Magnified image of the coupling capacitance. c, Magnified optical image of the dc-SQUID (superconducting quantum interference device) part and d, its scanning electron micrograph. The SQUID loop contains a three-Josephson-junction flux qubit, which is not used in this work.
Supplementary Figure 2: Resonant frequency $\omega_0^{PO}$ as a function of the flux bias $\Phi_{sq}$. Solid circles represent the experimental data, and the solid curve is a theoretical fit (Eq. 30) to the data for $|\Phi_{sq}/\Phi_0| \leq 0.32$. In the fitting, we assumed $L_{cav}$ of 1.08 nH, $C_{in}^{PO}$ of 15 fF, and $C_J$ of 50 fF, which are designed values. The deviation of the fitting outside the above range could be due to the loop inductance of the superconducting quantum interference device (SQUID), which is neglected in the theory and estimated to be $\sim 40$ pH by the simulation.
Supplementary Figure 3: Results of the numerical calculation. a, Example of the calculated $Q$ function. The parameters used in the calculation are $P_p/P_{p0} = 1.660$ (2.2 dB), $N^{PO} = 0.090$, $\theta_s = \pi/2$. The density matrix is truncated at $N = 80$, and the calculation is stopped at $\tau = 20$. b, Probability of $0\pi$ state as a function of the locking signal phase $\theta_s$ for $N^{PO} = 0.0090$ (blue), $N^{PO} = 0.090$ (green), and $N^{PO} = 0.90$ (magenta). Other parameters used in the calculation are the same as in a.
Supplementary Figure 4: Time trace of the output signal of the parametric phase-locked oscillator. $\omega_p/2\pi = 2 \times 10.193$ GHz, $t_p = 30$ ns, $t_r = 50$ ns, and $N^r = 5.5$ (See main article for definitions). The output signal at $\omega_p/2$ is down-converted to $\omega_{IF} = 2\pi \times 50$ MHz, and digitized at 1 GS/s. Qubit-control $\pi$ pulse is turned on ($t_c = 10$ ns). We selected 9250 traces which are in $1\pi$-phase state and averaged them (blue circles). Solid curve represents a fit to the data for $t > t_0 = 45$ ns. We assumed a function $V(t) = A(1 - e^{-(t-t_0)/\tau}) \cos(\omega_{IF}t + \phi_0)$, and obtained the time constant $\tau$ of 9.4 ns.
Supplementary Figure 5: Minimum probability of $1\pi$ state as a function of the locking-signal power. The non-zero minimum probability indicated by the horizontal dashed line is due to the initialization error of the qubit. $P_{S0}$ and $P_{S1}$ represent the microwave powers injected into the parametric phase-locked oscillator when the qubit is in $|0\rangle$ state and in $|1\rangle$ state, respectively.
Supplementary Figure 6: Histograms of the voltage of the reflected readout pulse. Qubit-control π pulse is turned off in a and turned on in b. The input power to the Josephson parametric amplifier (JPA) is $-122.5$ dBm, which is the same as in the Rabi-oscillation measurement in the main article. The JPA is operated at a gain of 13 dB and a bandwidth of 21 MHz with 1-dB-compression point of $-123$ dBm. The voltage was extracted from a 100-ns-long data sequence which was recorded after a delay of 450 ns from the beginning of the readout pulse in order to avoid a transient response of the JPA. This produces the large left peak corresponding to the qubit $|0\rangle$ state in b. The dashed lines represent the Gaussian fits to the distribution peaks to extract the mean values of $V_0 = -0.090$ V and $V_1 = 0.086$ V. Only right-hand side of the peak is fitted for b.
Supplementary Figure 7: Measurement of the qubit energy relaxation time. Probability of 0\textpi-state is plotted as a function of the delay time \(t_p\), between the readout and pump pulses with qubit-control \textpi-pulse on (green square) and off (blue circle). The pulse sequence is shown in Fig. 4a in the main article. Here, \(t_c = 10\) ns, \(t_r = 50\) ns, and \(t_d = 300\) ns. The solid curve is an exponential fit with a time constant of 690 ns.
Supplementary Note1: Experimental details

**Images of the Josephson parametric phase-locked oscillator.** Supplementary Fig. 1a shows the device image of the parametric phase-locked oscillator (PPLO). The device consists of a quarter wavelength (cavity length 2.6 mm) coplanar waveguide (CPW) resonator with a dc-SQUID (superconducting quantum interference device) termination and a pump line inductively coupled to the SQUID loop (mutual inductance $M \sim 1.0$ pH). The critical current of the Josephson junction of the SQUID is estimated to be 3.1 $\mu$A for each junction from the fitting shown in Supplementary Fig. 2. The device was fabricated by the planarized niobium trilayer process at MIT Lincoln laboratory. The resonator and the pump line are made of 150-nm-thick niobium film sputtered on a Si substrate covered by 500-nm-thick SiO$_2$ layer. Supplementary Fig. 1b shows the magnified image of the coupling capacitance between the microwave feedline and the resonator $C_{\text{PO}}$, which is designed to be 15 fF. Supplementary Figs 1c and d show the magnified optical image and the scanning electron micrograph of the dc-SQUID part, respectively.

**Error budget of the Rabi-oscillation contrast.** In the main article, we show Rabi oscillations with a contrast of 90.7%. Here, we present our analysis on the loss of the contrast. Possible sources of the error are (i) incomplete initialization of the qubit, (ii) insufficient power of the locking signal (LS), and (iii) qubit energy relaxation (including the gate error in the qubit-control $\pi$ pulse).

The error from the first source (incomplete initialization) is estimated from the direct measurement of the background qubit excitation by operating the device as Josephson parametric amplifier (JPA) [1], and found to be 2.6%. Half of this is equal to the non-zero minimum of the blue curve in Fig. 4c in the main article. This indicates that the error from the second source (non-locking error) is negligible at least when the qubit is in state $|0\rangle$.

Non-locking error is also estimated from the measurement result shown in Supplementary Fig. 5, which is similar to the measurement shown in Fig. 2e in the main article, and found to be negligible for both of the qubit states. Minimum probability of 1$\pi$ state decreases as we increase $P_s^{\text{PO}}$ and saturates at the level determined by the background qubit excitation when $P_s^{\text{PO}}$ is larger than $\sim -124$ dBm.

The microwave powers injected into the PPLO when the qubit is in $|0\rangle$ state and in $|1\rangle$ state are shown in the figure as $P_s^0$ and $P_s^1$, respectively. Because of the dispersive shift in $\omega_0^i$ and finite internal loss of the readout resonator, $P_s^1$ is slightly lower than $P_s^0$ which is set at $-122.5$ dBm. $P_s^1$ is estimated by directly measuring the amplitude of the reflected microwave when the qubit is excited to $|1\rangle$ state. Supplementary Fig. 6 shows the histograms of the voltage of the reflected readout pulse when the qubit-control $\pi$ pulse is not applied (a) and is applied (b) to the qubit. The reflected voltage is measured by a single-shot readout
using JPA [1]. The peak in Supplementary Fig. 6a corresponds to the qubit state $|0\rangle$, while the right peak in Supplementary Fig. 6b corresponds to the qubit state $|1\rangle$. By fitting the peaks with Gaussian functions, we extract the voltage of the reflected readout pulse when the qubit is in $|0\rangle$ state and in $|1\rangle$ state, which we call $V_0$ and $V_1$, respectively. $V_i$ and $P^i_s (i=0,1)$ satisfy the following relation

$$\frac{P^0_s G(P^0_s)}{P^1_s G(P^1_s)} = \left(\frac{V_0}{V_1}\right)^2, \quad (1)$$

where $G$ represents the power gain of the JPA. Note that $G$ is not necessarily independent of the input power when the input power is high, and we measured it independently (data not shown). From Eq. 1, $P^1_s$ is estimated to be $-123.0$ dBm.

The rest of the error 6.7% (1.0-0.907-0.026) is attributable to the third source (qubit relaxation). The energy relaxation time $T_1$ of the qubit is measured to be 690 ns (see below). Assuming $1 - \exp(-t_w/T_1) = 0.067$, $t_w = 48$ ns is obtained, which is close to the sum of $t_c$, $t_p$, and the response time of PPLO measured to be 10 ns (see Supplementary Fig. 4). Qubit relaxation during these times leads to a loss of Rabi-oscillation contrast.

**Qubit energy relaxation time.** Supplementary Fig. 7 shows the probability of $0\pi$ state as a function of $t_p$ when the qubit-control $\pi$ pulse is turned off and on. We fit the data ($\pi$ pulse on) from $t_p = 40$ ns to 3 $\mu$s with an exponential function, and obtain the time constant of 690 ns. This agrees with the qubit energy relaxation time $T_1$ of 694 ns obtained from an independent ensemble-averaged measurement using standard pulse sequence for $T_1$ measurement, namely, $\pi$ pulse followed by the delayed readout. Small decay of the data without $\pi$ pulse is possibly due to the qubit excitation induced by the readout pulse.
Supplementary Note 2: Theory and simulations

**Hamiltonian and equations of motion.** The Hamiltonian of the PPLO including a signal port for the locking signal (LS) and a fictitious loss port for internal loss of the resonator is given by [2]

\[
\mathcal{H}(t) = \mathcal{H}_{\text{sys}}(t) + \mathcal{H}_{\text{sig}} + \mathcal{H}_{\text{loss}},
\]

\[
\mathcal{H}_{\text{sys}}(t)/\hbar = \omega_0^{\text{PO}} \left[ a^\dagger a + \epsilon \cos(\omega_p t)(a + a^\dagger)^2 \right] + \gamma (a + a^\dagger)^4,
\]

\[
\mathcal{H}_{\text{sig}}/\hbar = \int dk \left[ \nu_k b_k^\dagger b_k + i \sqrt{\nu_k \kappa_1} \left( a^\dagger b_k - b_k^\dagger a \right) \right],
\]

\[
\mathcal{H}_{\text{loss}}/\hbar = \int dk \left[ \nu_k c_k^\dagger c_k + i \sqrt{\nu_k \kappa_2} \left( a^\dagger c_k - c_k^\dagger a \right) \right],
\]

where \(\omega_0^{\text{PO}}\) is the static resonant frequency of the PPLO, \(\omega_p\) and \(\epsilon\) represent the frequency and magnitude of the parametric modulation, respectively, \(\gamma\) represents the nonlinearity of the Josephson junction (JJ), \(a\) is the annihilation operator for the resonator, \(b_k\) \((c_k)\) is the annihilation operator for the photon in the signal (loss) port with a wave number \(k\) and a velocity \(\nu_k\) \((c_k)\), and \(\kappa_1\) \((\kappa_2)\) represents the coupling strength between the resonator and the signal (loss) port. The operators satisfy the following commutation rules: \([a, a^\dagger] = 1\), \([b_k, b_k^\dagger] = \delta(k - k')\), and \([c_k, c_k^\dagger] = \delta(k - k')\). The coupling constants \(\kappa_1\) and \(\kappa_2\) are related to the external and internal quality factors of the resonator as follows: \(\kappa_1 = \omega_0^{\text{PO}}/Q_0\) and \(\kappa_2 = \omega_0^{\text{PO}}/Q_1^{\text{PO}}\).

Below we consider the case where \(\omega_p = 2\omega_0^{\text{PO}}\).

From the Heisenberg equations of motion for \(b_k\), we obtain

\[
\frac{db_k(t)}{dt} = -i \nu_k b_k(t) - \sqrt{\nu_k \kappa_1} a(t).
\]

By solving this differential equation formally, we have

\[
b_k(t) = e^{-i \nu k t} b_k(0) - \sqrt{\nu_k \kappa_1} \int_0^t e^{-i \nu k (t - t')} a(t') dt'.
\]

We introduce the real-space representation of the waveguide field by \(\tilde{b}_r = (2\pi)^{-1/2} \int dk e^{i k r} b_k\). In this representation, the waveguide field interacts with the resonator at \(r = 0\) and the \(r < 0\) \((r > 0)\) region corresponds to the incoming \((\text{outgoing})\) field. From Eq. 7, we have

\[
\tilde{b}_r(t) = \tilde{b}_{r-\nu t}(0) - \sqrt{\frac{\kappa_1}{v}} \theta(r) \theta(t - r/v) a(t - r/v),
\]

where \(\theta(r)\) is the Heaviside step function. We define the input and output operators by

\[
\tilde{b}_{\text{in}}(t) \equiv \tilde{b}_{-0}(t) = \tilde{b}_{-\nu t}(0),
\]

\[
\tilde{b}_{\text{out}}(t) \equiv \tilde{b}_{+0}(t) = \tilde{b}_{\text{in}}(t) - \sqrt{\frac{\kappa_1}{v}} a(t).
\]
Using Eqs. 8 and 9, the field operator $\tilde{b}_0(t)$ at the resonator position ($r = 0$) is given by

$$\tilde{b}_0(t) = \frac{1}{\sqrt{2\pi}} \int b_k(t) dk = \tilde{b}_{in}(t) - \frac{1}{2} \sqrt{\frac{\kappa_1}{v}} a(t). \quad (11)$$

From the Heisenberg equations of motion for $a$, we obtain

$$\frac{da}{dt} = -i[H_{sys}(t), a] + \sqrt{\nu \kappa_1} \tilde{b}_0 + \sqrt{\nu \kappa_2} \tilde{c}_0. \quad (12)$$

Using Eq. 11 and its counterpart for $\tilde{c}_0$, Eq. 12 is rewritten as

$$\frac{da}{dt} = -i[H_{sys}(t), a] - \frac{\kappa}{2} a + \sqrt{\nu \kappa_1} \tilde{b}_{in}(t) + \sqrt{\nu \kappa_2} \tilde{c}_{in}(t), \quad (13)$$

where $\kappa = \kappa_1 + \kappa_2$. Now we switch to a frame rotating at $\omega_0^{PO}$ [namely, $a(t)e^{i\omega_0^{PO}t} \rightarrow a(t)$, $b_{in}(t)e^{i\omega_0^{PO}t} \rightarrow b_{in}(t)$, and $c_{in}(t)e^{i\omega_0^{PO}t} \rightarrow c_{in}(t)$] and drop the rapidly rotating terms in $H_{sys}(t)$. Then the static system Hamiltonian is given by

$$H_{sys}/\hbar = \frac{\epsilon_0^{PO}}{2}(a^2 + a^\dagger 2) + 6 \gamma a^\dagger a a. \quad (14)$$

Here, we neglected the term $12\gamma a^\dagger a$ which can be regarded as a small renormalization to $\omega_0^{PO}$. From Eqs. 13 and 14, we have

$$\frac{da}{dt} = -\frac{\kappa}{2} a - i\omega_0^{PO} \epsilon a^\dagger - 12i \gamma a^\dagger aa + \sqrt{\nu \kappa_1} \tilde{b}_{in}(t) + \sqrt{\nu \kappa_2} \tilde{c}_{in}(t). \quad (15)$$

Now we consider the classical amplitude of the resonator field, namely, $\langle a(t) \rangle$. We denote the locking signal applied from the signal port by $E_s(r, t) = E_s^* e^{i\omega_0^{PO}(r/v - t)}$. Then, we rigorously have $\langle \tilde{b}_{in}(t) \rangle = E_s^*$ and $\langle \tilde{c}_{in}(t) \rangle = 0$. Dividing $\langle a(t) \rangle$ into its quadratures as $\langle a(t) \rangle = q_x(t) - iq_y(t)$, their equations of motion are given by

$$\frac{dq_x}{dt} = -\frac{\kappa}{2} q_x + \omega_0^{PO} \epsilon q_y - 12 \gamma (q_x^2 + q_y^2) q_y + \sqrt{\kappa_1} |E_s| \cos \theta_s, \quad (16)$$

$$\frac{dq_y}{dt} = -\frac{\kappa}{2} q_y + \omega_0^{PO} \epsilon q_x + 12 \gamma (q_x^2 + q_y^2) q_x + \sqrt{\kappa_1} |E_s| \sin \theta_s, \quad (17)$$

where $E_s = |E_s| e^{i\theta_s}$, and we approximated $\langle a^\dagger a a \rangle$ to be $\langle a^\dagger \rangle \langle a \rangle^2$. Equations 16 and 17 can be recast as

$$\frac{dq_x}{dt} = -\frac{\kappa}{2} q_x + \frac{\partial g}{\partial q_y},$$

$$\frac{dq_y}{dt} = -\frac{\kappa}{2} q_y - \frac{\partial g}{\partial q_x}, \quad (19)$$

where

$$g(q_x, q_y) = \frac{\epsilon}{2} \omega_0^{PO} (q_y^2 - q_x^2) - 3\gamma (q_x^2 + q_y^2)^2 + \sqrt{\kappa_1} |E_s|((q_y \cos \theta_s - q_x \sin \theta_s). \quad (20)$$
Considering that $P_p / P_{p0} = (\epsilon / \epsilon_0)^2$, where $\epsilon_0 = \kappa / (2\omega_{\text{PO}})$ is the threshold for $\epsilon$, these are Eqs. 2 to 4 in the main article.

**Master equation.** We denote the resonator transition operator by $s_{mn} = |m\rangle \langle n|$, where $|m\rangle$ and $|n\rangle$ are the Fock states. Its Heisenberg equation is given, in the rotating frame $[s_{mn}(t)e^{i\omega_{\text{PO}}(n-m)t} \rightarrow s_{mn}(t)]$, by

$$
\frac{d}{dt}s_{mn} = \frac{i}{\hbar}[\mathcal{H}_{\text{sys}}, s_{mn}] + \frac{\kappa}{2}(2a^\dagger s_{mn}a - s_{mn}a^\dagger a - a^\dagger a s_{mn}) \\
+ \sqrt{\kappa_1}[s_{mn}, a^\dagger]b_{in}(t) - \sqrt{\kappa_1}b_{in}^\dagger(t)[s_{mn}, a] \\
+ \sqrt{\kappa_2}[s_{mn}, a^\dagger]c_{in}(t) - \sqrt{\kappa_2}c_{in}^\dagger(t)[s_{mn}, a].
$$

(21)

where the static system Hamiltonian $\mathcal{H}_{\text{sys}}$ is given by Eq. 14. Using again that $\langle \tilde{b}_{in}(t) \rangle = E_{a}^* \text{ and } \langle \tilde{c}_{in}(t) \rangle = 0, \langle s_{mn} \rangle$ evolves as

$$
\frac{d}{dt} \langle s_{mn} \rangle = \frac{i\omega_{\text{PO}}}{2}(m(m-1)\langle s_{m-2,n} \rangle + (m+1)(m+2)\langle s_{m+2,n} \rangle \\
- \sqrt{n(n-1)}\langle s_{m,n-2} \rangle - \sqrt{(n+1)(n+2)}\langle s_{m,n+2} \rangle) \\
+ 6i\gamma[m(m-1) - n(n-1)]\langle s_{mn} \rangle + \frac{\kappa}{2}[2\sqrt{m(m+1)n(n+1)}\langle s_{m+1,n+1} \rangle - (m+n)\langle s_{mn} \rangle] \\
+ \sqrt{\kappa_1}E_a(\sqrt{n}\langle s_{m,n-1} \rangle - \sqrt{m+1}\langle s_{m+1,n} \rangle) \\
- \sqrt{\kappa_1}E_a^\dagger(\sqrt{n+1}\langle s_{m,n+1} \rangle - \sqrt{m-1}\langle s_{m-1,n} \rangle),
$$

(22)

where we have used $a^\dagger s_{mn}a = \sqrt{(m+1)(n+1)}s_{m+1,n+1}$ and similar equalities.

Since $\langle s_{mn} \rangle = \text{Tr}[\rho s_{mn}] = \rho_{nm}$, Eq. 22 is equivalent to the following master equation,

$$
\frac{d\rho}{dt} = -\frac{i}{\hbar}[\mathcal{H}_{\text{int}}, \rho] + \frac{\kappa}{2}(2a^\dagger \rho a - a^\dagger \rho a - \rho a^\dagger a),
$$

(23)

where

$$
\mathcal{H}_{\text{int}}/\hbar = \frac{\epsilon_{\text{PO}}}{2}(a^2 + a^\dagger a^\dagger) + 6\gamma a^\dagger a^\dagger aa + i\sqrt{\kappa_1}|E_a| (e^{i\theta_a}a^\dagger - e^{-i\theta_a}a).
$$

(24)

By introducing dimensionless time $\tau = t\kappa/2$, Eq. 23 becomes

$$
\frac{d\rho}{d\tau} = -i[\mathcal{H}_{\text{int}}, \rho] + (2a^\dagger \rho a - a^\dagger \rho a - \rho a^\dagger a).
$$

(25)

Here the dimensionless Hamiltonian $\mathcal{H}_{\text{int}}'$ is given by

$$
\mathcal{H}_{\text{int}}' = \mathcal{H}_{\text{int}}/(\hbar\kappa/2) = \gamma' a^\dagger a^\dagger aa + \frac{1}{2}\sqrt{\frac{P_p}{P_{p0}}}(a^2 + a^\dagger a^\dagger) + i\sqrt{N_{\text{PO}}} (e^{i\theta_a}a^\dagger - e^{-i\theta_a}a),
$$

(26)

where $\gamma' = 12\gamma / \kappa$, $\sqrt{N_{\text{PO}}} = \sqrt{\kappa_1}|E_a|/(\kappa/2)$. 
We numerically solve Eq. 25 by expanding $\rho$ in the number state basis \[3\], namely $\rho = \sum_{m,n=0}^{N} \rho_{mn} |m\rangle \langle n|$, and calculate the $Q$ function $Q(z) = \langle z|\rho|z\rangle/\pi$, where $z$ is a coherent state \[2\].

**Experimental parameters.** To simulate the experiments, we need to determine the parameters such as $N^{PO}$, $P_{p0}$, and $\gamma$. $N^{PO}$ is determined from the above definition and $|E_s| = \sqrt{P_{p0}/\hbar \omega_{0}^{PO}}$. Since we do not precisely know the mutual inductance between the pump line and the SQUID loop in the PPLO, we can only roughly determine $P_{p0}$, the threshold for the pump power, from the measurement shown in Fig. 2a in the main article, and leave it as a semi-adjustable parameter. $\gamma$ is calculated based on the theory in Ref. 4. It is related with the wavenumber $k$ of the first mode of the CPW resonator terminated by a SQUID,

$$\gamma = -\left(\frac{2\pi}{\Phi_0}\right)^2 \frac{\hbar B_k}{8 C_k}, \quad (27)$$

where

$$B_k = \frac{(1/4) \cos^2(kd)}{1 + 2kd/\sin(2kd)}, \quad (28)$$

$$C_k = \frac{C_{cav}}{2} \left[1 + \frac{\sin(2kd)}{2kd}\right] + C_J \cos^2(kd). \quad (29)$$

Here, $d$ and $C_{cav}$ are the length and the total capacitance of the CPW resonator, respectively, and $C_J$ is the total junction capacitance of the SQUID. The flux dependent resonant frequency is given by

$$\omega_0^{PO}(\Phi_{sq}) = \frac{1}{\sqrt{L_k(C_k + C_{in}^{PO})}}, \quad (30)$$

where

$$1/L_k = \frac{(kd)^2}{2L_{cav}} \left[1 + \frac{\sin(2kd)}{2kd} + \frac{2C_J}{C_{cav}} \cos^2(kd)\right]. \quad (31)$$

Here, $L_{cav}$ is the total inductance of the CPW resonator, and $kd$ is determined by the equation \[4\].

$$kd \tan(kd) = \left(\frac{2\pi}{\Phi_0}\right) L_{cav} 2I_0 \cos\left(\frac{\Phi_{sq}}{\Phi_0}\right) - \frac{C_J}{C_{cav}} (kd)^2, \quad (32)$$

where $I_0$ represents the critical current of each of the SQUID JJ.

We fit the data shown in Supplementary Fig. 2 by Eq. 30 with fitting parameters of $I_c$ and $C_{cav}$, which are determined to be 3.1 $\mu$A and 410 fF, respectively. Using Eqs. 27 to 29, and 32, we calculate $\gamma$ to be $-2.3 \times 10^5$ Hz at $\omega_0^{PO}/2\pi = 10.51$ GHz.

**Simulation of non-locking error.** Supplementary Fig. 3a shows an example of the calculated $Q$ function. The parameters used in the calculation are $P_p/P_{p0} = 1.660$ (2.2 dB), $N^{PO} = 0.090$, $\theta_s = \pi/2$. The density
matrix is truncated at $N = 80$, which we confirmed large enough for these parameters. Also, the calculation is stopped at $\tau = 20$, which we confirmed long enough for the system to become stationary. We clearly observe two distribution peaks, which correspond to $0\pi$ and $1\pi$ states of PPLO. Based on this result, we get the probability of $0\pi$ state by integrating $Q$ for $q_x > 0$. Supplementary Fig. 3b shows the probability of $0\pi$ state as a function of the LS phase for different $N^{PO}$'s. $N^{PO}$'s are chosen from those in Fig. 2d in the main article. The agreement is fairly good.
Supplementary References


