Supplementary Note 1

Class-A to phase equation

We start with the class-A laser equation
\[ \frac{dE}{dt} = (1 + i\alpha) (1 - |E|^2) E + Y + i\Delta E + \eta e^{-i\Omega} E, \]  
(1)

By defining \( E = re^{i\phi} \) we get
\[ \frac{dr}{dt} = \left( 1 - r^2 \right) \rho + Y \cos \phi + \eta \rho r \cos (\Delta \phi - \Omega), \]  
(2)
\[ \frac{d\phi}{dt} = \alpha \left( 1 - r^2 \right) + \Delta - \frac{Y}{\rho} \sin \phi + \eta \frac{\rho r}{\rho} \sin (\Delta \phi - \Omega), \]  
(3)

where we defined \( \Delta \phi = \phi_r - \phi \) and \( \phi_r = \phi(t - \tau) \). Assuming that \( Y, \eta, \) and \( \Delta \) are of order \( \varepsilon \), and inserting a multiple time scale expansion as
\[ \rho(t_1, \varepsilon t_2) = 1 + \varepsilon \tau (t_1, \varepsilon t_2) + \mathcal{O}(\varepsilon^2), \]  
(4)
\[ \phi(t_1, \varepsilon t_2) = \phi (t_1, \varepsilon t_2), \]  
(5)
\[ \frac{d}{dt} = \frac{\partial}{\partial t_1} + \varepsilon \frac{\partial}{\partial t_2}, \]  
(6)

we find that the phase does not depend on the fast time scale, i.e. \( \partial \phi/\partial t_1 \equiv 0 \), while at first order we obtain
\[ \frac{\partial r}{\partial t_1} = -2\varepsilon \tau + Y \cos \phi + \eta \cos (\Delta \phi - \Omega), \]  
(7)
\[ \frac{\partial \phi}{\partial t_2} = -2\alpha \varepsilon \tau + \Delta - \frac{Y}{\rho} \sin \phi + \eta \sin (\Delta \phi - \Omega). \]  
(8)

We can now perform the adiabatic elimination of the fast variable \( r \) on the time scale \( t_1 \) and find that it is slaved to the slow variations of \( \phi \) on the time scale \( t_2 \) as
\[ 2\varepsilon \tau = Y \cos \phi + \eta \cos (\Delta \phi - \Omega). \]  
(9)

Upon replacing the expression of \( \varepsilon \) we get
\[ \frac{1}{\sqrt{1 + \alpha^2}} \frac{d\phi}{dt_2} = \frac{\Delta}{\sqrt{1 + \alpha^2}} - Y \sin (\phi + u) + \eta \sin (\Delta \phi - \Omega - u), \]  
(10)
\[ \sqrt{1 + \alpha^2} \]  
(11)

where we defined \( u = \arctan \alpha \). Finally, rescaling the time and the delay as \( (t', \tau') = (t, \tau)Y\sqrt{1 + \alpha^2} \) which incidentally redefines the detuning and the feedback rate as \( \Delta' = \Delta / (Y\sqrt{1 + \alpha^2}) \) and \( \chi = \eta / Y \) yields:
\[ \frac{d\phi}{dt'} = \Delta' - \sin (u + \phi) + \chi \sin (\phi_r - \phi - \Omega - u). \]  
(12)

A last possible simplification amounts in redefining the phase \( \theta = u + \phi \) and introducing \( \psi = \Omega + u \) reducing the problem to
\[ \frac{d\theta}{dt'} = \Delta' - \sin \theta + \chi \sin (\theta_r - \theta - \psi). \]  
(13)

The phase model contains only three parameters, \( \Delta' \) the strength of the detuning with respect to the injection, the feedback rate and its phase, the delay being mostly irrelevant as long as it is large compared to the internal time scale of Eq. (13). The saddle-node steady states \( \bar{\theta} \) and \( \bar{\psi} \) are defined as \( \bar{\theta} = \arcsin (\Delta' - \chi \sin \psi) \) and \( \bar{\psi} = \pi - \bar{\theta} \).

Supplementary note 2

Sine-Gordon spatio-temporal equation

Assuming that the solution is almost of period \( \tau' \) we get that \( \theta_r, \tau' - \tau \ll 1 \) hence, we can expand Eq. (13) as
\[ \frac{d\theta}{dt'} = (\Delta' - \sin \theta) - \chi \sin \psi + \chi \cos \psi (\theta_r - \theta) \]  
(14)
\[ + \frac{\chi \sin \psi}{2} (\theta_r - \theta)^2 + \cdots \]  
(15)

Similarly to the analysis close to an Andronov-Hopf bifurcation presented in [1], we introduce two time scales \( u_1 = \varepsilon t' \) and \( u_2 = \varepsilon^2 t' \), the chain rule being \( dt' = \varepsilon du_1 + \varepsilon^2 du_2 \).

The central point in our analysis lies in the way we expand the delayed contribution as \( \theta_r = \theta (\varepsilon u_1 + \varepsilon^2 u_2 - \varepsilon^2 \tau') \).

Doing so, we assume that the solution evolves slowly from one round-trip toward the next, which is described by the variations on the slow time \( u_2 \), up to a small drift term \( \varepsilon v \) whose contribution is accounted for in the variable \( u_1 \). Expanding the delayed term up to second order leads to
\[ \frac{\partial \theta}{\partial t_2} = \left( \varepsilon v \frac{\partial \theta}{\partial u_1} + \frac{(\varepsilon v)^2}{2} \frac{\partial^2 \theta}{\partial u_1^2} - \varepsilon^2 \tau' \frac{\partial \theta}{\partial u_2} \right), \]  
(16)
\[ (\theta_r - \theta)^2 \sim (\varepsilon v)^2 \left( \frac{\partial \theta}{\partial u_1} \right)^2 + \mathcal{O}(\varepsilon^3), \]  
(17)

which eventually yields the modified Sine-Gordon equation with a tilt as well as a quadratic velocity contribution
\[ \frac{\partial \theta}{\partial \xi} = \sin \bar{\theta} \sin \theta + \frac{\partial^2 \theta}{\partial x^2} + \tan \psi \left( \frac{\partial \theta}{\partial x} \right)^2. \]  
(18)

In Eq. (18), we removed the drift velocity induced by the feedback term by identifying that \( v = \sqrt{\chi \cos (\psi)} \).

In addition, we rescaled the slow time by defining \( \xi = u_2 \varepsilon^{-2} / (1 + \tau' \chi \cos \psi) \), scaled the pseudo-space as \( x = u_1 \varepsilon^{-1} \sqrt{2\chi \cos \psi} \) and defined the equilibrium value \( \bar{\theta} \) with \( \sin \bar{\theta} = \Delta' - \sin \theta \).
Supplementary note 3

Superposition coefficient with the incoming field

We will use the methodology developed in [2] which amount in solving exactly the wave propagation within the linear empty regions of the VCSEL. Our starting point is the scalar Maxwell equation in the temporal Fourier representation

\[
(\partial_z^2 + \nabla_r^2) E(\omega, r) + \frac{\omega^2}{c^2} n^2 E(\omega, r) = \frac{-\omega^2}{c^2 \varepsilon_0} P(\omega, r),
\]

(19)

Assuming that the transverse profile of index guiding defines the modal structure of the resonator, we can located at \( r \) where we assumed that the Quantum-wells (QWs) are

\[
\text{two reflections and transmissions at the top and bottom yielding the four boundary conditions that consist in the presence of an antinode of the field on the QW which selects incidentally only an even number of } \frac{\pi}{l} \text{ oscillations.}
\]

Since the numerator of \( F_1 \) is the strongly varying function we only expand \( 1 - r_1 r_2 e^{iql} \) and neglect the small variations of \( F_2 \). Imposing that \( q \) is close to resonance \( \omega_0 \) for which \( \exp(2iq_0L) = \exp(2iq_0 (l = \frac{1}{2})) = 1 \), we obtain

\[
1 - r_1 r_2 e^{iql} \sim (1 - r_1 r_2) - r_1 r_2 (q - q_0) 2iLc t_1 e^{iql} \left(1 + r_2 e^{2iq(L-l)} \right) \sim t_1 (-1)^{m} (1 + r_2) + \cdots
\]

(32)

where the length \( L_c = L - \frac{1}{2} \left( \partial_Q \ln r_2 + \partial_Q \ln r_1 \right) \) takes into account the frequency dependence of \( r_j(\omega) \), i.e. the frequency dependent penetration depth into the DBRs. Expanding \( q \) around \( q_0 = n\omega_0/c \) saying that \( \omega \rightarrow \omega_0 + i\delta \) yields the final field evolution equation as

\[
\tau \frac{dE}{dt} = \frac{i\omega_0 (1 + r_1) (1 + r_2)}{2n\varepsilon_0 c r_1 r_2} P - \frac{1 - r_1 r_2}{r_1 r_2} E \\
+ t_1 \frac{1 + r_2}{r_1 r_2} (-1)^{m} \tilde{Y}.
\]

(33)

As a last step, we must evaluate the field at the laser output \( A \) as a combination of the reflection of the injected beam \( r_1 \tilde{Y} \) as well as transmission of the left intracavity propagating field \( L_- \), i.e.

\[
A = t_1 L_- + r_1 \tilde{Y} \\
= \frac{t_1 e^{iqL}}{1 + r_1 e^{2iqL}} \left(1 + r_2 e^{2iqL} \right) \tilde{Y} r_1 + e^{2iqL} \left(1 + r_1 e^{2iqL} \right)
\]

(35)

where we used the Stokes relation \( t't - r'r = 1 \) and \( r' = -r \) to simplify the last equation. Around resonance such expression simplifies into

\[
A = E \frac{t_1 (-1)^m}{1 + r_1} - \tilde{Y}
\]

(36)

The \((-1)^m \) is irrelevant in the sense that the QW field experiences also an injection field with a \((-1)^m \) meaning that the superposition is always in anti-phase and can be absorbed into a redefinition of \( E \rightarrow -E \), hence in the following we will assume \( m \) even. Introducing the adiabatic expression for the gain as \( P = g(a-i)(N-N^*) \) as well as the carrier equation of evolution, scaling time
to the photon lifetime of device $\kappa = \frac{1 - r_1 r_2}{r_1 r_2}$ and redefining the injected field as $Y = h\tilde{Y}$ with $h = t_1' (1 + r_2)/(r_1 r_2)$ yields

$$
\dot{E} = [(1 + i\alpha) D - 1] E + Y,
\dot{T} D = J - (1 + |E|^2) D,
A = \frac{t_1}{1 + r_1} (E - k Y) \sim E - k Y.
$$

with the following definition of $k$ as

$$
k = \frac{1 - r_1 r_2}{(1 - r_1)(1 + r_2)},
$$

where we used again the Stokes relation, defined $T = \kappa/\gamma_{\parallel}$ and normalized the carriers as $D = \Gamma g \omega_0 (2n\epsilon_0)^{-1} (N - N^*)$.

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