SUPPLEMENTARY INFORMATION

I. A BRIEF INTRODUCTION TO TOPOLOGICAL PHOTONICS

In this part of Supplementary Information we briefly introduce the idea of topological photonics. Currently there have been many reviews and tutorials on this subject, here we refer to ref. [1] for the recent advances in topological photonics and ref. [2, 3] for the pedagogical tutorial of topological band theory.

**Topological band theory.**—Topology considers the properties of objects preserved under continuous deformations such as stretching and bending, and quantities remaining invariant under such continuous deformations are denoted as topological invariants. For instance, closed surface has the topological invariant genus which counts the number of the holes within the surface. Notice that such topological invariant does not respond to small perturbations as long as the holes are not created by tearing or removed by gluing. Two objects which looks very different can be equivalent in the sense of topology, and objects with different topologies can be classified into different equivalent classes by their topological invariants: The sphere and the spoon are equivalent because both of their genus are zero, while the torus and the coffee cup are also topological equivalent as their genus are one.

The application of the topological ideas in photonic systems can be traced back to the exciting developments of topological band theory and the discovery of topological insulators in condensed matter physics [4]. The bulk of topological insulators behave as an insulator while electrons can move along the edge without dissipation or back-scattering even in the presence of defect. The first example of topological insulators is the integer quantum Hall effect (IQHE) discovered in 1980 [5] of two-dimensional (2D) electrons in a uniform external magnetic field. It is demonstrated later that the quantized Hall conductance can be represented by the nontrivial topology of the bulk band structure in reciprocal space [6]. Such nontrivial topology is described by the topological invariant called Chern number, which is defined as

\[
C_h = \frac{1}{2\pi} \oint_{BZ} \nabla_k \times \langle u_h(k) | i\nabla_k | u_h(k) \rangle,
\]

with \(h\) the index of energy band and \(u_h(k)\) the Bloch function of momentum \(k\) in the first Brillouin zone (FBZ). Such topological invariant is insensitive to the local perturbation (e.g. disorder and defects), and can only be changed through the closing and reopening of the
band gaps. It is further shown that an IQHE system with $q$ bands and $q - 1$ gaps can be described by Bloch functions on a complex energy surface with genus $q - 1$ corresponding to the $q - 1$ gaps of the lattice [7]. It is exactly this topological invariant that leads to the quantization of the measured Hall conductance with fluctuations less than $10^{-9}$ and insensitive to factors including sample sizes, compositions, and defects.

A direct consequence of the nontrivial topological band structure is the emergence of edge state modes (ESMs). The existence of the ESMs can be explained by the following simple argument [1, 2, 8]: we put two topological insulators, each with different topological band structure, close to each other, so that they share a boundary and away from the boundary the two insulators extend to $\pm \infty$, respectively. As we know, the topology of the bands cannot be changed unless the bulk gaps collapse and reopen again, we then come to the conclusion that these two insulators can not be connected trivially at the boundary region, and there must be a topological phase transition happened some where closing and reopening the band gaps, i.e., it must have ESMs crossing the band gaps. Otherwise, the bands of the two insulators can be connected smoothly in a gapped way, which by default would mean that the topology in the whole space would be the same. This argument applies to a boundary between any two topologically different insulators, and the presence of these gapless ESMs thus serves as an unambiguous signature of the topological non-triviality of the bulk band structure. The gapless spectra of the ESMs are topologically protected, which means that their existence is guaranteed by the difference of the topologies of the bulk materials on the two sides, and their propagation is robust against the perturbations in the materials including disorders and defects.

**Topological photonics.**—Topological photonics is the application of topological band theory in photonic systems [1]. As the theory of topological insulators considered mainly the electronic systems in the single-particle picture [2], the existing results of characterizing the band topologies can be directly applied to interaction-free bosonic systems, and there have been numerous theoretical and experimental investigations of the photonic analogue of quantum Hall effects in the context of various artificial photonic metamaterials during the past few years [9–12]. The motivation of topological photonics lies in both the sides of experimental application and fundamental physics. In the systems of integrated optical circuits, back-reflection induced by disorder and defects is a major source that hinders the transmission of information. Therefore, it is expected that the unidirectional ESMs, if
synthesized, can be used to transmit electromagnetic waves without back-reflection even in the presence of arbitrarily large disorder. Closely mimicking the topologically protected quantized Hall conductance of 2D electronic materials, such ideal transport may offer novel designs and functionalities for photonic systems with topological immunity to fabrication errors or environmental changes.

On the other hand, topological photonics also brings new physics of fundamental importance. As the band topology of spinless particles remains trivial as long as the time-reversal symmetry is preserved [2], the effective magnetic field has to be synthesized for the photons in the metamaterials [11–13], which is represented by the nontrivial hopping phases on the lattice through Peierl’s substitution. In condensed matter physics, the currently-achievable strong magnetic field exposed to conventional electronic materials are \( \sim 45 \text{T} \) in the continuous regime and \( \sim 80 \text{T} \) in the pulsed regime [14, 15], which result in the electrons’ magnetic lengths far more larger than the lattice spacing. Namely, in order to acquire the one-flux-quantum penetration (i. e. an Aharonov-Bohm phase \( 2\pi \)), an electron has perform Lorentz circulation with a very large round loop, covering hundreds or even tens of thousand of unit-cells. Meanwhile, based on the fabrication and manipulation of artificial photonic metamaterials, it is expected to obtain arbitrarily large Aharonov-Bohm phase for photons in a loop containing only few unit-cells, indicating that the synthetic magnetic field for photons is by several orders larger than those for electronic materials. In addition, it can be obviously noticed that the photonic systems is naturally non-equilibrium and bosonic, exhibiting neither chemical potential nor fermionic statistics. This is in stark contrast to the electronic materials. From the above two points of view, topological photonics is not only the simple photonic analog of quantum Hall effects, but also a realm of exotic but less-explored fundamental physics.

Meanwhile, the transfer from electronics to photonics also imposes several challenges in theory and experiment. An obvious and important issue is how to measure the topological invariants. For electronic systems, the topological invariant of the bands is measured directly through the quantized Hall conductance in transport experiments. This method is nevertheless not applicable in photonic systems because of the absence of the fermionic statistics. In this manuscript, this problem is overcome through the realization of the gedanken adiabatic pumping process proposed by R. B. Laughlin [8, 16]: We consider a two-dimensional lattice with a uniform perpendicular magnetic field \( \phi \), as shown in Fig. S1.
We place periodic boundary conditions in the $y$-direction but open edges in the $x$-direction. Due to the periodic boundary condition in the $y$-direction, a thought experiment can be devised in which the sample is wrapped in the $y$-direction into a cylinder, with $x$ being parallel to the axis of the cylinder. Therefore, the uniform magnetic flux $\phi$ becomes radial to the cylinder and ESMs emerge at the two edges. Through the cylinder and parallel to the $x$-axis, a magnetic flux $\alpha$ is inserted. The flux $\alpha$ can shift the momentum of the ESMs. When exactly one flux quanta is threaded through the Laughlin cylinder, the ESM spectrum return to its original form with an integer number of ESMs been transferred. This integer is the winding number of the ESMs which is equivalent to the Chern number of the bulk bands. Here we remark that the probing of topological invariants of our scheme is based exactly on this idea: Notice that the spatial configuration of the proposed lattice in Fig. 1(b) of Main Text is topologically equivalent to the Laughlin cylinder in Fig. S1(a), i. e. they both have genus one. The frequencies of the single- or multi-site pumping are exploited as the Fermi level to select the ESMs we are interested in, and the desired information of those modes can be extracted from the dependence of the steady-state photon numbers on the location of the pumping sites and pumping frequency.

![Diagram of Laughlin cylinder](image)

**FIG. S1.** Sketch of the Laughlin cylinder.
II. EIGENMODES OF THE LATTICE AND THEIR COUPLING

In this part of Supplementary Information, we analyze in detail the eigenmodes of the lattice and the coupling between them induced by the grounding SQUIDs. These issues can be best illustrated by the investigation of the highlighted four-TLR unit-cell shown in Fig. 1(a) of Main Text. We first derive the well-localized eigenmodes of the unit-cell and then study the coupling between them. During this investigation, we estimate various parameters of the proposed circuit and verify several assumptions we have made in Main Text based on recently reported experimental data of parametric coupling in circuit QED [17–19]. As we focus solely on the highlighted unit-cell, the influence from the other part of the lattice is omitted by setting the inductances of the four grounding SQUIDs at the individual ends as infinitesimal.

The eigenmodes of the unit-cell.—The common grounding SQUID of the four TLRs (labeled 1, 2, 3, and 4) in Fig. 1(a) of Main Text is characterized by its effective Josephson energy $E_J = E_{J0} \cos(\pi \Phi_{\text{ext}}/\Phi_0)$ with $E_{J0}$ being the maximal Josephson energy, $\Phi_{\text{ext}}$ being the external flux bias, and $\Phi_0 = h/2e$ being the flux quantum. Due to its very small inductance, this SQUID can be regarded as a low-voltage shortcut of the four TLRs, and it is this boundary condition that allows the definition of the individual TLR modes [20, 21]. Physically speaking, a particular TLR (e.g. the TLR 1) can hardly “feel” the other three because the currents from them will flow mostly to the ground through the SQUID without entering it. The lowest eigenmodes of the lattice can thus be approximated by the $\lambda/2$ modes of the TLRs.

• Model. More rigorously, the Lagrangian of the unit-cell can be written as

$$\mathcal{L} = \sum_j \int_0^{L_j} dx \left[ \frac{1}{2} \left( c \left( \frac{\partial \phi_j(x,t)}{\partial t} \right) \right)^2 - \frac{1}{l} \left( \frac{\partial \phi_j(x,t)}{\partial x} \right)^2 \right] + \frac{1}{2} C_J \dot{\phi}_j^2 + E_J \cos \left( \frac{\phi_j}{\phi_0} \right)$$

$$\approx \sum_j \int_0^{L_j} dx \left[ \frac{1}{2} \left( c \left( \frac{\partial \phi_j(x,t)}{\partial t} \right) \right)^2 - \frac{1}{l} \left( \frac{\partial \phi_j(x,t)}{\partial x} \right)^2 \right] + \frac{1}{2} C_J \dot{\phi}_j^2 - \frac{1}{2L_j} \phi_j^2$$

with $c/l$ the capacitance/inductance per unit length of the TLRs, $j = 1, 2, 3, 4$ the label of the four TLRs, $L_j$ the length of the $j$th TLR, $C_J$ the capacitance of the SQUID,
\[ \phi_0 = \Phi_0 / 2\pi \] the reduced flux quantum, \( L_J = \phi_0^2 / E_J \) the Josephson inductance of the SQUID, \( V_j(x, t) \) the voltage distribution on the \( j \)th TLR, \( \phi_j(x, t) = \int_{-\infty}^t dt' V_j(x, t') \) the corresponding node flux distribution, \( V_i(t) \) the voltage across the grounding SQUID, and \( \phi_J(t) = \int_{-\infty}^t dt' V_i(t') \). In deriving equation (3), we have linearized the grounding SQUID by assuming \( E_J \cos(\phi_J/\phi_0) \approx -\phi_J^2 / 2L_J \). This assumption proves to be consistent with the calculation performed in the latter of this section.

Based on Euler-Lagrangian equation we get the equation of motion of \( \phi_j(x, t) \) in the bulk of the TLRs

\[
\frac{\partial^2 \phi_j}{\partial x^2} - v^2 \frac{\partial^2 \phi_j}{\partial t^2} = 0,
\]

with \( v = 1/\sqrt{d} \), while from Kirchhoff’s law we obtain the boundary conditions

\[
\phi_j(x = 0) = 0, \quad \phi_j(x = L_j) = \phi_j, \quad -\frac{1}{L_J} \sum_j \left. \frac{\partial \phi_j}{\partial x} \right|_{x=L_j} = \frac{\phi_J}{L_J} + C_J \ddot{\phi}_J.
\]

Equation (4) can be solved by the variable separation ansatz \( \phi_j(x, t) = \sum_m f_{j,m}(x) g_m(t) \) where \( m \) is the index of the eigenmode and \( f_{j,m} \) is the node flux function of the \( m \)th eigenmode in the \( j \)th TLR. Exploiting equation (5) we have \( f_{j,m}(x) = C_{j,m} \sin(k_m x) \), and by inserting \( f_{j,m}(x) \) into equation (7) we get the transcendental equations

\[
\sum_{j'} C_{j',m} L_J k_m \cos(k_m L_{j'})
+ \left( 1 - \frac{C_J L_J k_m^2}{c l} \right) C_{j,m} \sin(k_m L_j) = 0,
\]

for \( j = 1, 2, 3, 4 \) which completely determine \( f_{j,m}(x) \) up to a normalization constant.

- **The eigenmodes: numerical calculation.** We solve equation (8) numerically and plot in Fig. S2 the normalized node flux distributions of the four lowest eigenmodes. Here we choose the orthonormality relation between the four eigenmodes as [22–24]

\[
\sum_{j'} \int_0^{L_{j'}} dx f_{j',m}(x) f_{j',n}(x)
+ \frac{C_J}{c} f_{j,m} (L_j) f_{j,n} (L_j) = \delta_{mn}.
\]
The circuit parameters are selected based on recent experiments of dynamic Casimir effect and parametric conversion in circuit QED \([17–19, 25, 26]\). For the TLRs, we choose the capacitance per unit length as \(c = 1.6 \times 10^{-10} \text{ F} \cdot \text{m}^{-1}\), the inductance per unit length as \(l = 4.08 \times 10^{-7} \text{ H} \cdot \text{m}^{-1}\), and the lengths \([L_1, L_2, L_3, L_4] = [6.16, 5.35, 4.24, 3.84] \text{ mm}\ [25, 26]\), while for the grounding SQUID, we set the capacitance as \(C_J = 0.5 \text{ pF}\), the maximal critical current as \(I_{J0} = E_{J0}/\phi_0 = 75.5 \mu\text{A}\), and the d. c. bias \(\Phi_{\text{ext}} = \Phi_{\text{dc}} = 0.37\Phi_0\ [17, 18, 27, 28]\). These settings result in the effective critical current \(I_J = E_J/\phi_0 = 30 \mu\text{A}\), \(\omega_0/2\pi = 10 \text{ GHz}\), and \(\Delta/2\pi = 1.5 \text{ GHz}\). From Fig. S2 we can find out that the eigenmodes are well-localized in the corresponding TLRs, and the one-to-one correspondence can thus be established between each of the TLRs and each of the eigenmodes. Such separation property can be further quantified by the energy storing ratio (ESR) of the \(m\)th mode defined as

\[
\text{ESR}_m = E_m^m/E_m, \tag{10}
\]

with

\[
E_m^m = \int_0^{L_m} \text{d}x \frac{1}{2} [c\omega_m^2 + \frac{1}{l}k_m^2]f_{m,m}^2(x), \tag{11}
\]

\[
E_m = \sum_j \int_0^{L_j} \text{d}x \frac{1}{2} [c\omega_m^2 + \frac{1}{l}k_m^2]f_{j,m}^2(x)
+ \frac{1}{2} [C_J\omega_m^2 + \frac{1}{L_J}]f_{m,m}^2(x = L_m), \tag{12}
\]

and \(\omega_m = vk_m\). It can be directly found out that the ESR factor of the \(m\)th mode represents the energy stored in the \(m\)th TLR versus the energy of the whole mode. The ESR factors of the four eigenmodes versus \(I_J\) is shown in Fig. S3. We notice that the ESR factors increase with increasing \(I_J\). When \(I_J\) approaches the manuscript-chosen 30 \(\mu\text{A}\), these four factors are all above 98.5\%, indicating the well-separation of the eigenmodes.

- **Canonical quantization.** The quantization of the eigenmodes is then straightforward. With the orthonormality relation in equation (9) the Lagrangian in equation (3) can be simplified as

\[
\mathcal{L} = \sum_m \frac{c\dot{g}_m^2}{2} - \frac{c\omega_m^2g_m^2}{2}, \tag{13}
\]
FIG. S2. Normalized node flux distributions of the lowest four eigenmodes in the four-TLR unit-cell with $I_J = 30 \, \mu A$ (in unit of $m^{-1}$).

FIG. S3. Localization of the four eigenmodes quantified by the ESR factors versus $I_J$.

and the corresponding Hamiltonian can be further be derived as

$$\mathcal{H}_0 = \sum_m \frac{\pi_m^2}{2c} + \frac{c\omega_m^2 g_m^2}{2} \quad (14)$$

with $\pi_m = \partial \mathcal{L} / \partial \dot{g}_m$ being the canonical momentum of $g_m$. Through the definition of the creation/annihilation operators

$$a_m^\dagger = \sqrt{\frac{\omega_m c}{2\hbar}} g_m - i \sqrt{\frac{1}{2\hbar \omega_m c}} \pi_m, \quad (15)$$

$$a_m = \sqrt{\frac{\omega_m c}{2\hbar}} g_m + i \sqrt{\frac{1}{2\hbar \omega_m c}} \pi_m, \quad (16)$$
\( \mathcal{H}_0 \) can finally be written as

\[
\mathcal{H}_0 = \sum_m \hbar \omega_m (a_m^\dagger a_m + \frac{1}{2}).
\]  \hspace{1cm} (17)

**Verification of several assumptions.**—Based on the parameters chosen above, here we can go back to check the several approximations and assumptions we have made in Main Text.

- **Estimation of \( \phi_J \).** We can write \( \phi_J \) as

\[
\phi_J = \sum_j \phi_j^j (a_j + a_j^\dagger),
\]  \hspace{1cm} (18)

with

\[
\phi_j^j = f_{j,j}(x = L_j) \sqrt{\frac{\hbar}{2\omega_j c}}
\]  \hspace{1cm} (19)

being the r. m. s node flux fluctuation of the \( j \)th mode across the grounding SQUID. Based on the parameters chosen previously, we can calculate

\[
(\phi^1, \phi^2, \phi^3, \phi^4)/\phi_0 = (1.8, 2.0, 2.6, 2.7) \times 10^{-3}.
\]  \hspace{1cm} (20)

Such small fluctuation of \( \phi_J \) indicates that the eigenmodes derived in equation (17) can be regarded as the individual \( \lambda/2 \) modes of the TLRs slightly mixed by the grounding SQUID with small but finite inductance (see also Fig. S2).

- **D. C. mixing induced by the SQUID.** We can then estimate to what extent the finite inductance of the grounding SQUID mixes the individual \( \lambda/2 \) modes of the TLRs. We recall that such mixing can be physically traced back to the d. c. Josephson coupling energy

\[
\mathcal{E}_{dc} = -E_J \cos \left( \frac{\phi_J}{\phi_0} \right)
\]

\[
\approx \frac{1}{2} \left( \frac{\phi_J}{\phi_0} \right)^2 E_{J0} \cos \left( \frac{\Phi_{dc}^{ex}}{2\phi_0} \right).
\]  \hspace{1cm} (21)

By representing \( \phi_J \) as the form shown in equations (18) and (19), we have

\[
\mathcal{E}_{dc} = \sum_{m,n} T_{m,n}^{dc} (a_m^\dagger + a_m)(a_n^\dagger + a_n),
\]  \hspace{1cm} (22)
with
\[ T_{d,c}^{m,n} = \frac{\phi^m \phi^n}{\phi_0^2} E_{J0} \cos \left( \frac{\Phi_{d,c}^{ex}}{2\phi_0} \right). \]  

(23)

\( T_{d,c}^{m,n} \) can then be regarded as the d. c. mixing between the individual \( \lambda/2 \) modes induced by the static bias of the grounding SQUID. With the chosen parameters above, we have the estimation
\[ T_{d,c}^{m,n}/2\pi \in [60, 90] \text{ MHz} \approx [0.04, 0.06] \Delta/2\pi, \]

(24)

with the estimated \( \Delta/2\pi = 1.5 \text{ GHz} \). This estimation is in consistence with the previous presentation that the grounding SQUID slightly mixes the \( \lambda/2 \) modes of the TLRs.

**Assumption of linear expansion.** Based on the smallness of the \( \phi^j / \phi_0 \), we can estimate the higher fourth order nonlinear term of \(-E_J \cos(\phi^j / \phi_0)\) as
\[ E_{d,c}^4 \approx \frac{1}{48} \left( \frac{\phi^j}{\phi_0} \right)^4 E_{J0} \cos \left( \frac{\Phi_{d,c}^{ex}}{2\phi_0} \right) \in 2\pi \left[ 10^{-2}, 10^{-1} \right] \text{ kHz} \approx 10^{-6} T_{d,c}^{m,n}, \]

(25)
i. e. six orders of magnitude smaller than the reserved second-order terms in equations (3) and (21). Such small term can then be safely neglected, and the validity of the Taylor expansion in Main Text and in deriving equation (3) is therefore verified.

**Estimation of the non-nearest coupling.** With the estimation of the d. c. nearest-neighbor mixing induced by the grounding SQUID, we can further consider the non-nearest-neighbor coupling induced by the grounding SQUIDs. For a particular TLR, e. g. the TLR 1, its current will flow mostly through the grounding SQUID except a small fraction entering the other three TLRs. This small but nonzero fraction will flow further through the individual grounding SQUIDs of those three TLRs, causing the non-nearest-neighbor coupling between distant TLRs. However, due to the described current-division mechanism \([29, 30]\), the strengths of the non-nearest-neighbor coupling decay exponentially versus distances between the TLRs. With the proposed parameters, the strength of the next-nearest-neighbor coupling can be estimated as
\[ T_{NNN} \approx T_{r,r'}^{d,c} \frac{L_{j}}{L_{j}} \in 2\pi \left[ 0.4, 0.8 \right] \text{ MHz} \]
\[ \approx [10^{-2}, 10^{-1}] T, \]

(26)
with the estimated $T/2\pi \in [10, 15]$ MHz (see the following estimation). Such small scale perturbation cannot close and reopen the band gaps which are of the order $T$ and consequently cannot influence significantly the synthesisization and detection of the topological-protected ESMs.

**Parametric coupling between the eigenmodes.**—The parametric coupling between the four eigenmodes originates from the dependence of $E_J$ on $\Phi_{\text{ext}}$

$$E_J = E_{J0} \cos \left( \frac{1}{2\phi_0} (\Phi_{\text{ex}}^{dc} + \Phi_{\text{ex}}^{ac}(t)) \right)$$

$$\approx E_{J0} \cos \left( \frac{\Phi_{\text{ex}}^{dc}}{2\phi_0} - \frac{E_{J0}\Phi_{\text{ex}}^{ac}(t)}{2\phi_0} \sin \left( \frac{\Phi_{\text{ex}}^{dc}}{2\phi_0} \right) \right),$$

where we have assumed that a small a. c. fraction $\Phi_{\text{ex}}^{ac}(t)$ has been added to $\Phi_{\text{ext}}$ with $|\Phi_{\text{ex}}^{ac}(t)| \ll |\Phi_{\text{ex}}^{dc}|$ such that the first order expansion at the d. c. bias point $\Phi_{\text{ex}}^{dc}$ can be performed.

- **The three-tone pulse and the consequent parametric coupling.** We thus set that $\Phi_{\text{ex}}^{ac}(t)$ is composed of three tones as

$$\Phi_{\text{ex}}^{ac}(t) = \Phi_{32}\cos(2\Delta t - \theta_{32}) + \Phi_{14}\cos(4\Delta t + \theta_{14})$$

$$+ \Phi_h\cos(\Delta t - \theta_h),$$

where the $2\Delta/4\Delta$ tones are exploited to induce the vertical $2 \leftrightarrow 3/4 \leftrightarrow 1$ hoppings respectively, and the $\Delta$ tone is used for the horizontal $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ hoppings [21]. By representing $\phi_j$ as the form shown in equations (18) and (19) we obtain the a. c. coupling from the second term of equation (21)

$$\mathcal{H}_{AC} = \frac{E_{J0}\Phi_{\text{ex}}^{ac}(t)}{4\phi_0} \sin \left( \frac{\Phi_{\text{ex}}^{dc}}{2\phi_0} \right) \left( \sum_j \phi_j^2 \left( a_j + a_j^\dagger \right) \right)^2.$$  

In the rotating frame with respect to $\mathcal{H}_S$, the induced parametric photon hopping can be derived from equation (30) as

$$\mathcal{H}_T = e^{i\mathcal{H}_S} \mathcal{H}_{AC} e^{-i\mathcal{H}_S}$$

$$\approx [\mathcal{T}_{32} e^{i\theta_{32}} a_3^\dagger a_2 + \mathcal{T}_{14} e^{i\theta_{14}} a_1^\dagger a_4$$

$$+ \mathcal{T}_h e^{i\theta_h} (a_2 a_1 + a_4^\dagger a_3)] + \text{h.c.},$$

(31)
where $\mathcal{T}_j^{ac}$ are the effective hopping strengths proportional to the corresponding $\Phi_j$ in $\Phi_{ac}^{ex}(t)$, and the fast-oscillating terms in $e^{iH_0}H_{AC}e^{-iH_0}$ are omitted by the rotating wave approximation. A further inspection finds out that the nontrivial vertical hopping phases are synthesized because the $4 \leftrightarrow 1$ and $2 \leftrightarrow 3$ links can be independently controlled by the $2\Delta$ and $4\Delta$ tones, while the horizontal hopping phases are leaved trivial due to the same $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$ transition frequency $\Delta$. Such configuration is suitable for the implementation of Landau gauge in Main Text, i.e.

$$A = [0, A_y(x), 0], \tag{32}$$

$$B = Be_z = \left[0, 0, \frac{\partial}{\partial x}A_y(x)\right], \tag{33}$$

where only the vertical hopping phases play the essential role. With the already-existed d. c. bias of the grounding SQUID set as $\Phi_{dc}^{ex}/\Phi_0 = 0.37$, $I_{10} = 75.5 \mu$A, and $I_J = 30 \mu$A, we further choose the amplitude of the three tones as $[18, 19, 31, 32]$

$$[\Phi_{32}, \Phi_{14}, \Phi_h] = \Phi_0 [3.5\%, 3.8\%, 2.6\%]. \tag{34}$$

In this situation, the homogeneous coupling strength $\mathcal{T}/2\pi = 10 \text{ MHz}$ of the $1 \leftrightarrow 2$, $2 \leftrightarrow 3$, $3 \leftrightarrow 4$, and $4 \leftrightarrow 1$ couplings can be induced.

- **Verification of the plasma frequency of the grounding SQUID.** Here we should be careful that the modulating frequency of $\Phi_{dc}^{ac}(t)$ is limited by the plasma frequency of the grounding SQUID $\omega_p = \sqrt{8E_cE_J}$, beyond which the internal degrees of freedom of the SQUID can be activated and complex quasi-particle excitations will emerge [20]. Meanwhile, this requirement is fulfilled in our scheme by the very small inductance of the grounding SQUID. With the parameters selected in this section, we have the estimation

$$E_c/2\pi = \frac{e^2}{4\pi C_J} = 38.7 \text{ MHz}, \quad E_J/2\pi = 1.5 \times 10^4 \text{ GHz}, \tag{35}$$

and consequently

$$\omega_p/2\pi = 68 \text{ GHz} \approx 45\Delta/2\pi, \tag{36}$$

for the chosen $\Delta/2\pi = 1.5 \text{ GHz}$, indicating the effective suppression of the unwanted excitation of the grounding SQUID.
Refinement: suppressing the unwanted imperfections. Here we consider some minor revisions of the proposed parametric conversion method which help us significantly suppress the unwanted imperfections discussed in Main Text. The fabrication error includes the deviations of the realized circuit parameters from the ideal settings (e.g., the lengths and the unit capacitances or inductances of the TLRs) leading to the disorder $\delta \omega$ of the eigenmodes’ frequencies. Meanwhile, with developed microelectronic techniques such fabrication-induced disorder can be pushed to the level of $10^{-4}$ \cite{33} which corresponds to $\delta \omega_r \approx 10^{-4} \omega_r \sim 10^{-1} T$. Moreover, one can slightly adjust the frequencies of the three-tone parametric conversion pulses in the grounding SQUIDS according to the fabrication-induced frequency shift. With such adjustment the fabrication-induced diagonal disorder can be effectively cancelled while the performance of the parametric conversion scheme is not affected.

The effect of next-nearest-neighbor coupling relies critically on the frequency match between the next-nearest-neighbor TLRs. To suppress its effect we can use the two-sublattice strategy \cite{34}: We keep the eigenfrequencies of the highlighted four-TLR unit-cells shown in Fig. 1(a) of Main Text invariant, but shift the frequencies of its neighboring four-unit cells by a small amount, e.g. to $2\pi \left[ 9.3, 10.3, 12.3, 16.3 \right]$ GHz, respectively. Such modification does not influence the performance of the proposed parametric conversion method as we merely need to adjust the modulating frequencies of the grounding SQUIDS accordingly. However, the effective next-nearest-neighbor photon hopping which is estimated to be of the order $2\pi \left[ 0.4, 0.8 \right]$ MHz is significantly suppressed by the 300 MHz energy difference between the next-nearest-neighbor TLRs in the neighboring four-TLR unit-cells.

III. LOW FREQUENCY NOISE OF THE LATTICE

In this part of Supplementary Information, we analyze the influence of the low-frequency $1/f$ noise on the proposed circuit \cite{35}. In actual experimental circuits, the $1/f$ noise at low frequencies far exceeds the thermodynamic noise. Understanding this fluctuation is thus crucial for our scheme. We first describe the phenological Dutta-Horn model of the low-frequency $1/f$ noise following Ref. \cite{36} and then estimate the effect produced by various kinds of $1/f$ noise based on previously reported experimental data. Our estimations show
that the $1/f$ noise induces diagonal and off-diagonal disorder which are both too small to destroy the topological properties of the ESMs. The feasibility of our scheme is thus pinpointed from this point of view by its topological robustness against the low-frequency $1/f$ fluctuations.

The random telegraph noise model of the $1/f$ noise.—It is generally believed that a noise $\delta O(t)$ of the physical variable $O$ in solid-state physics exhibiting the $1/f$ spectrum can be modelled by the summation of random telegraph noises (RTNs) emitted from an ensemble of bistable fluctuators [36]. The bistable fluctuators can be defects on the substrate trapping and releasing itinerant electrons (the charge noise), pieces of magnetic flux jumping into and out of a SQUID loop (the flux noise), or switches in the Josephson junction opening and closing Josephson supercurrent channels (the critical current noise).

- **Spectrum of a single RTN.** A classical random telegraph noise $\xi(t)$ is a Poissonian fluctuator switching abruptly between two values $-v$ and $v$ with transition rate $\gamma$. Its correlation function has the form

$$\langle \xi(t)\xi(0) \rangle = v^2 e^{-2\gamma|t|}, \quad (37)$$

and the corresponding spectrum takes the form

$$S_\xi(\omega) = \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \xi(t)\xi(0) \rangle = \frac{4v^2\gamma}{\omega^2 + 4\gamma^2}. \quad (38)$$

- **An RTN ensemble leading to $1/f$ noise.** We next consider $\delta O(t)$ as the summation of a large number of independent bistable noise fluctuators, i.e.

$$\delta O(t) = \sum_j \xi_j(t), \quad (39)$$

with each $\xi_j(t)$ taking its own $v_j$ and $\gamma_j$. The $1/f$ noise is obtained when the switching rate $\gamma_j$ depends exponentially on a parameter $l_j$ as

$$\gamma_j = \gamma_0 e^{-l_j/l_0}, \quad (40)$$

with $l_j$ taking the uniform distribution

$$P(l_j) = P_0 = \frac{1}{l_{\text{max}} - l_{\text{min}}}. \quad (41)$$
in the range \([l_{\text{min}}, l_{\text{max}}]\). The switching rate \(\gamma_j\) thus has the distribution

\[
P(\gamma_j) = \frac{P_0 l_0}{\gamma_j},
\]

in the range \([\gamma_{\text{min}}, \gamma_{\text{max}}]\), with \(\gamma_{\text{min}/\text{max}} = \gamma_0 e^{-l_{\text{min/\text{max}}}/l_0}\). The total noise spectrum of the ensemble, given by the form

\[
S_\delta(\omega) = N \int dv P(v) v^2 \int_{\gamma_{\text{min}}}^{\gamma_{\text{max}}} \frac{d\gamma}{\omega^2 + 4\gamma^2}.
\]

with \(N\) being the total number of the fluctuators, takes the asymptotic behavior

\[
S_\delta(\omega) \propto \begin{cases} 
\text{const.}, & \text{if } \omega \ll \gamma_{\text{min}}, \\
1/\omega, & \text{if } \gamma_{\text{min}} \ll \omega \ll \gamma_{\text{max}}, \\
1/\omega^2, & \text{if } \gamma_{\text{max}} \ll \omega, 
\end{cases}
\]

i. e. the 1/f type spectrum emerges in the deep middle of \([\gamma_{\text{min}}, \gamma_{\text{max}}]\). We can further write \(S_\delta(\omega)\) as

\[
S_\delta(\omega) = 2\pi A_O^2 \omega,
\]

where \(A_O\) labels the noise spectrum of \(\delta O\) at \(2\pi \times 1\) Hz, taking the same dimension of \(O\).

The relation in equation (40) leading to the 1/f spectrum is ubiquitous in solid-state physics. For instance, for a particle trapped in a double-well potential, the tunneling rate through the potential barrier depends exponentially on both the height and the width of the barrier. A flat distribution of distances or barrier heights, which is very likely to happen in disordered solid-state systems, thus leads to 1/f noise. Another example is the thermally activated tunneling or trapping with rate expressions of the form \(\gamma_0 e^{-E/k_B T}\), where \(E\) denotes the activation energy or the depth of a trapping potential. From this point of view, in the following calculation we set

\[
\gamma_{\text{min}}/2\pi = 1\ \text{Hz, } \gamma_{\text{max}}/2\pi = 1\ \text{GHz}.
\]

These upper and lower limits are chosen based on the scale of the experiment time and the \(\sim 50\) mK temperature scale of the dilute refrigerator, respectively [37–39].
Due to its low frequency property, we can treat the $1/f$ noises as quasi-static, i.e., the noises do not vary during an experimental run, but vary between different runs. The variance of the fluctuating $\delta O$ can then be evaluated from $S_O(\omega)$ as

$$
\langle (\delta O(t))^2 \rangle = \frac{1}{2\pi} \int d\omega \int_{-\infty}^{+\infty} dt e^{i\omega t} \langle \delta O(t) \delta O(0) \rangle \\
= \frac{1}{2\pi} \int d\omega S_O(\omega) \approx A_O^2 (\ln \gamma_{\text{max}} - \ln \gamma_{\text{min}}),
$$

indicating that the range of the fluctuating $\delta O$ can be roughly estimated as $S_O(\omega) \in [-5, 5] A_O$.

$1/f$ noise in the proposed circuit.—The $1/f$ noise existing in superconducting quantum circuits can generally be traced back to the fluctuations of three degrees of freedom, namely, the charge, the flux, and the critical current. In the following we estimate their strength and their influence on the proposed scheme.

- **Charge noise.** Experiments on single-electron-tunneling devices have shown that the magnitude of background charge noise is typically $A_q \approx 10^{-3} e$ for a junction with area $0.01 \ \mu m^2$ [40, 41]. Meanwhile, the circuit considered consists of TLRs which are linear elements and grounding SQUIDs which have very small charging energies due to their big capacitances. Therefore, the proposed circuit is insensitive to the charge noise. Such insensitivity roots in the same origin of the charge-insensitive capacitance-shunted transmon qubit which has been extensively investigated in Ref. [39].

- **Flux noise.** We next consider the flux noise penetrated in the loops of the grounding SQUIDs. Previously, various measurements on flux noise have shown that $A_\Phi/\Phi_0 \in [10^{-6}, 10^{-5}]$ does not vary greatly with the loop size, inductor value, or temperature [42–45]. Therefore the strength of the flux noise $\delta \Phi$ can be estimated as $\delta \Phi/\Phi_0 \in [10^{-5}, 10^{-4}]$. Such fluctuation is by two orders of magnitude smaller than the d.c. $\Phi_{\text{dc}} = 0.37 \Phi_0$ and the a.c. amplitudes $\Phi_{32}, \Phi_{14}, \Phi_{h} \in [2.6\%, 3.5\%] \Phi_0$.

The existence of the flux noise $\delta \Phi$ shifts the d.c. bias $\Phi_{\text{dc}}$ of the grounding SQUID in a quasi-static way. Its consequence can then be evaluated through the perturbative Taylor expansion of equations (21) and (30) with respect to $\Phi_{\text{dc}}$ as

$$
\delta E_{\text{dc}} \approx \frac{\delta \Phi}{4\phi_0^3} E_{\text{dc}} \sin \left( \frac{\Phi_{\text{dc}}}{2\phi_0} \right) \left[ \sum_j \phi^j (a_j + a_j^\dagger) \right]^2,
$$

(48)
Based on the chosen parameters in Main Text and in Supplementary Information, we can evaluate that the fluctuating $\delta \Phi$ causes

\[
\delta \omega_r / 2\pi \in [10^{-3}, 10^{-2}] \text{ MHz} < 10^{-3} \mathcal{T} / 2\pi, \\
\delta \mathcal{T}_{rr} / 2\pi \in [10^{-4}, 10^{-3}] \text{ MHz} < 10^{-4} \mathcal{T} / 2\pi,
\]

from equations (48) and (49) and based on the estimated $\mathcal{T} / 2\pi = 10$ MHz. These flux-noise-induced diagonal and off-diagonal fluctuations are both much smaller than the band gaps which are of the order $\mathcal{T}$ (see Fig. 2 of Main Text) and the spectral spacing between the ESM peaks which are of the order $10^{-1} \mathcal{T}$ (see Fig. 5 of Main Text). Such small fluctuations can neither destroy the topological properties of the ESM by closing the band gaps nor mix the resolution of the ESMs in the steady state photon number measurement. Therefore, we come to the conclusion that our scheme can survive in the presence of the $1/f$ flux noise.

- **Critical current noise.** Experiments have shown that the critical current noise has $\mathcal{A}_{I_{J0}} \approx 10^{-6} I_{J0}$ for a junction at temperature 4 K [42, 44, 46]. The parameter $\mathcal{A}_{I_{J0}} / I_{J0}$ proves to be proportional to the temperature down to at least 100 mK. Therefore we set $\mathcal{A}_{I_{J0}} / I_{J0} \in [10^{-7}, 10^{-6}]$. Similar to the flux noise, the influence of the critical current noise can also be estimated by the Taylor expansion of $E_{dc}$ and $\mathcal{H}_{AC}$ with an alternative respect to $E_{J0} = I_{J0} \hbar / 2e$. Following the estimation similar to that of the previous flux noise, we can evaluate that the fluctuating $\delta I_{J0}$ causes

\[
\delta \omega_r / 2\pi \in [10^{-4}, 10^{-3}] \text{ MHz} < 10^{-4} \mathcal{T} / 2\pi \\
\delta \mathcal{T}_{rr} / 2\pi \in [10^{-5}, 10^{-4}] \text{ MHz} < 10^{-5} \mathcal{T} / 2\pi.
\]

These effects can be safely neglected because they are even smaller than the flux-noise-induced effects. Actually, This estimation is consistent with the experimental demonstration that the dominant noise source in a large-junction Josephson phase qubit is the flux noise and not the junction critical-current noise [44].


